

Predictions for Gromov-Witten invariants of noncommutative resolutions

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In this paper, we apply recent methods of localized GLSM's to make predictions for Gromov-Witten invariants of noncommutative resolutions, as defined by *e.g.* Kontsevich, and use those predictions to examine the connectivity of the SCFT moduli space. Noncommutative spaces, in the present sense, are defined by their sheaves, their B-branes. Examples of abstract CFT's whose B-branes correspond with those defining noncommutative spaces arose in examples of abelian GLSM's describing branched double covers, in which the double cover structure arises nonperturbatively. This note will examine the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$, which realizes this phenomenon. Its Landau-Ginzburg point is a noncommutative resolution of a (singular) branched double cover of \mathbb{P}^3 . Regardless of the complex structure of the large-radius $\mathbb{P}^7[2, 2, 2, 2]$, the Landau-Ginzburg point is always a noncommutative resolution of a singular space, which begs the question of whether the noncommutative resolution is connected in SCFT moduli space by a complex structure deformation to a smooth branched double cover. Using recent localization techniques, we make a prediction for the Gromov-Witten invariants of the noncommutative resolution, and find that they do not match those of a smooth branched double cover, telling us that these abstract CFT's are not continuously connected to sigma models on smooth branched double covers.

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1 Introduction

This short note describes a prediction for Gromov-Witten invariants of an abstract CFT, interpreted in [1] as a ‘noncommutative resolution’ (in Kontsevich’s sense) of a singular branched double cover, and its application to determine connectivity of the moduli space of such CFT’s.

The noncommutative resolutions in question are defined mathematically just by their sheaf theory. That by itself does not suffice to define a CFT; instead, the work [1] found examples of gauged linear sigma models (GLSMs) which, at special points, RG flowed to abstract CFT’s with the same B-branes as those defined by noncommutative resolutions. For that reason, those abstract CFT’s were identified with physical realizations of the noncommutative resolutions. (This also means that this notion of noncommutative geometry has a different physical realization than previous realizations of noncommutative geometry in physics such as [2, 3].)

One purpose of this note is simply to make a prediction for Gromov-Witten invariants of the noncommutative resolutions appearing at the Landau-Ginzburg point of the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$. Now, it is not clear that a noncommutative resolution, as defined mathematically, should admit a notion of Gromov-Witten invariants, as they need not come with *e.g.* symplectic structures, but here we have a full-fledged CFT whose B-branes coincide

with those of the noncommutative resolution, so one can hope to apply the extra structure implicit in the CFT. That said, since they are abstract CFT's, which might not have an analogue of a large-radius limit (as suggested by monodromy computations in [1]), it is still not in principle obvious that Gromov-Witten invariants can be defined. Nevertheless, using the methods of [4], we are able to compute Gromov-Witten invariants of these theories.

Another application of those Gromov-Witten invariants is to test the connectivity of the SCFT moduli space. If the noncommutative resolutions in question are connected in SCFT moduli space to smooth branched double covers via a complex structure deformation, then A model correlation functions and Gromov-Witten invariants of the noncommutative resolutions should match those of smooth branched double covers. However, it is not clear that the moduli spaces need be so connected. For example, the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$ always generates noncommutative resolutions at the Landau-Ginzburg point, never smooth branched double covers, and conversely starting from a smooth branched double cover it is not clear physically what marginal operator would be responsible for generating a noncommutative resolution. As a result, it is not *a priori* obvious that the noncommutative resolutions and smooth branched double covers lie on the same SCFT moduli space.

When we compute Gromov-Witten invariants, we find that the curve counting in the noncommutative resolution is different from that in a corresponding smooth branched double cover, hence, as we suggested above, the two components of the SCFT moduli space cannot be connected through a complex structure deformation. Apparently the noncommutative resolutions arising at the Landau-Ginzburg point of the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$ are examples of ‘frozen’ singularities, in the sense that the singularity and noncommutative resolution cannot be deformed away.

We begin in section 2 by reviewing pertinent aspects of the analysis of the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$, and the appearance of noncommutative resolutions. As a warm-up, in section 3 we compute Gromov-Witten invariants for $\mathbb{P}^7[2, 2, 2, 2]$ using the methods of [4]. These Gromov-Witten invariants are known, and we recover the standard result. Finally, in section 4 we use the same methods to compute Gromov-Witten invariants at the Landau-Ginzburg point of the same model, which is interpreted as a noncommutative resolution of a singular branched double cover.

2 Review of pertinent GLSM's

Up until several years ago, it was thought that all gauged linear sigma models:

- Can only describe geometries that are global complete intersections,
- Those geometries can be realized physically only as the critical locus of a superpotential,

and

- All geometric phases of GLSM's are birational to one another.

The papers [5, 6, 1] found counterexamples to all of the statements above, in both nonabelian [5, 6] and abelian [1] GLSM's. The nonabelian examples produced Pfaffian and determinantal varieties, realized by strongly-coupled gauge theoretic effects (and more recently, realized perturbatively [7, 8]), and the abelian examples gave (noncommutative resolutions of) branched double covers, also realized via nonperturbative effects. These branched-double-cover structures have been independently checked in *e.g.* [7][section 6.5] using gauge-theoretic dualities and [9] using an analysis of D-brane probes and matrix factorizations. Although the different phases in these examples are not birational, they are instead related by 'homological projective duality' [10, 11, 12], and more recent work [13] strongly supports the assertion that all phases of GLSM's are related by homological projective duality. See [14, 15, 16] for reviews of the abelian GLSM examples and branched double covers.

The 'noncommutative resolutions' referred to above, are the focus of this article. Mathematically, these are certain generalized notions of spaces defined by their sheaf theory, broadly speaking; see [11] for the specific pertinent noncommutative resolutions, and *e.g.* [17, 18, 19, 20, 21, 22] for closely related material. To define a CFT associated to a noncommutative resolution, one needs more data than just a set of sheaves. What was uncovered in [1] are a set of abstract CFT's, which look 'mostly' like branched double covers, away from singularities, and which everywhere possess B-branes matching those defining the noncommutative resolution. For that reason, matching B-branes, the abstract CFT's were identified with a physical realization of noncommutative resolutions, a result which also matched a mathematical prediction of homological projective duality.

Let us quickly review the structure of the abelian GLSM's in which branched double covers arose, beginning with an example in which no noncommutative resolution was present. The simplest example discussed in [1] was the GLSM for the complete intersection Calabi-Yau $\mathbb{P}^3[2, 2]$. The superpotential for this theory is of the form

$$W = \sum_a p_a G_a(\phi) = \sum_{ij} \phi_i \phi_j A^{ij}(p),$$

where the ϕ 's act as homogeneous coordinates on \mathbb{P}^3 , the G_a 's are the two quadrics, and A^{ij} is a symmetric 4×4 matrix with entries linear in the p 's, determined by the G_a 's.

At the Landau-Ginzburg point of this theory, where the p_a are not all zero, the superpotential acts as a mass matrix for ϕ 's. Naively, this is problematic: we are left with a theory containing only p 's, which looks like a sigma model on \mathbb{P}^1 , which cannot possibly be Calabi-Yau. However, a closer analysis reveals subtleties. First, since the p 's are charge 2, there is a trivially-acting \mathbb{Z}_2 here (technically, a \mathbb{Z}_2 gerbe structure), which physics interprets [23] as a double cover. Second, the mass matrix $A^{ij}(p)$ has zero eigenvalues along the degree four

hypersurface $\{\det A = 0\}$. With a bit of further analysis discussed in [1], one argues that this flows in the IR to a nonlinear sigma model on a branched double cover of \mathbb{P}^1 , branched over a degree four hypersurface – an example of a Calabi-Yau. In fact, both $\mathbb{P}^3[2, 2]$ and the branched double cover are elliptic curves.

Analogous analyses apply to many other examples. The next simplest involves the GLSM for $\mathbb{P}^5[2, 2, 2]$, which is a K3 surface. Its Landau-Ginzburg point is interpreted as a branched double cover of \mathbb{P}^2 , branched over a degree six locus, which is another K3.

The case after that is more interesting. The Landau-Ginzburg point of the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$ is, naively, a branched double cover of \mathbb{P}^3 . However, mathematically that branched double cover is always singular, and yet the GLSM behaves as if it is describing a smooth space. The resolution described in [1] is that the GLSM is instead describing a ‘noncommutative resolution’ of the singular branched double cover. That structure can be seen most directly in matrix factorizations in a Landau-Ginzburg model intermediate in RG flow. The noncommutative resolution is defined by its sheaf theory, and specifically, its sheaves are all sheaves of B -modules over \mathbb{P}^3 (equivalently, sheaves of modules over Azumaya algebras over the branched double cover), for B a sheaf of even parts of Clifford algebras defined by the GLSM superpotential. Matrix factorizations in the Landau-Ginzburg model automatically have this structure, hence we can identify the Landau-Ginzburg model with a physical realization of the noncommutative resolution.

In this paper we shall apply the ideas of [4, 24] to predict the Gromov-Witten invariants of noncommutative spaces, such as those discussed in [1].

3 $\mathbb{P}^7[2, 2, 2, 2]$: large-radius analysis

Let us consider the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$. This GLSM has two sets of fields: Φ_i , $i \in \{1, \dots, 8\}$, with gauge $U(1)$ charge 1 and $U(1)_V$ charge $2\mathbf{q}$, and P_a , $a \in \{1, \dots, 4\}$, with gauge $U(1)$ charge -2 and $U(1)_V$ charge $2 - 4\mathbf{q}$. Following the discussion in [4], the partition function¹ for this theory for $r \gg 0$ is [26, 27]

$$Z_{nlsm} = \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-4\pi i r \sigma} (Z_\Phi)^8 (Z_P)^4,$$

where

$$Z_\Phi = \frac{\Gamma(\mathbf{q} - i\sigma - m/2)}{\Gamma(1 - \mathbf{q} + i\sigma - m/2)}, \quad Z_P = \frac{\Gamma((1 - 2\mathbf{q}) + 2i\sigma + 2m/2)}{\Gamma(2\mathbf{q} - 2i\sigma + 2m/2)}.$$

¹ In principle, one expects that the expression above will pick up a phase as one wanders around on the SCFT moduli space and the phases of the localizing supercharges vary [25], hence the particular expression above corresponds to a particular choice of normalization.

As in [4], define $\tau = \mathbf{q} - i\sigma$, so that the partition function above becomes

$$Z_{nlsm} = e^{-4\pi\mathbf{q}r} \sum_{m \in \mathbb{Z}} e^{-i\theta m} \int_{\mathbf{q}-i\infty}^{\mathbf{q}+i\infty} \frac{d\tau}{2\pi i} e^{4\pi r \tau} \left(\frac{\Gamma(\tau - m/2)}{\Gamma(1 - \tau - m/2)} \right)^8 \left(\frac{\Gamma(1 - 2\tau + m)}{\Gamma(2\tau + m)} \right)^4. \quad (1)$$

We shall first consider the $r \gg 0$ region in the Kähler moduli space, and verify that the method of [4] recovers the Gromov-Witten invariants of $\mathbb{P}^7[2, 2, 2, 2]$. Then, we shall apply the same method to the $r \ll 0$ phase to make a prediction for Gromov-Witten invariants of the noncommutative resolution of the branched double cover.

When $r \gg 0$, we can close the contour above on the left half-plane. Assume $0 < \mathbf{q} < 1/2$.

Let us find poles that will contribute to the contour integral above.

First, we claim that there will be no net contribution from Z_P , in the sense that all poles in the numerator are cancelled out by corresponding poles in the denominator. (The rest may have zeroes at some of the zeroes of Z_Φ , on the other hand.) First, note $\Gamma(1 - 2\tau + m)$ has poles at

$$\tau = \frac{1}{2}(m + 1 + k)$$

for $k \geq 0$, and these will lie inside the contour when $k + 1 < -m$. Since $k \geq 0$, this can only happen if $m < 0$, in which case, $0 \leq k < -(m + 1)$. Similarly, $\Gamma(2\tau + m)^{-1}$ has zeroes when

$$\tau = -\frac{1}{2}(m + k)$$

for $k \geq 0$, and these will lie inside the contour when $-m - k \leq 0$, *i.e.* $k \geq \max\{0, -m\}$. In particular, any pole of $\Gamma(1 - 2\tau + m)$, defined by some k , is matched by a zero of $\Gamma(2\tau + m)^{-1}$ at

$$\tau = -\frac{1}{2}(m + (-2m - 1 - k)).$$

As a consistency check, note that since $k < -m - 1$,

$$-2m - 1 - k > -2m - 1 + m + 1 = -m$$

hence (as $m < 0$) $-2m - 1 - k$ is in the right range to define a zero of $\Gamma(2\tau + m)^{-1}$.

Now that we have established that Z_P will not contribute to the pole count, let us turn to Z_Φ . The numerator $\Gamma(\tau - m/2)$ has poles at

$$\tau = m/2 - k$$

for $k \geq 0$, and these will lie inside the integration contour when $k \geq m/2$.

In the case of Z_P , all poles were matched by zeroes, but for Z_Φ , only some of the poles will be matched by zeroes. The remaining unmatched poles will be counted by $k \geq \max\{0, m\}$. We can see this as follows. The zeroes of $\Gamma(1 - \tau - m/2)^{-1}$ are located at

$$\tau = 1 - m/2 + k_1$$

for $k_1 \geq 0$. A zero coincides with a pole when

$$1 - m/2 + k_1 = m/2 - k$$

or equivalently

$$k_1 = m - k - 1 \geq 0,$$

which requires $k \leq m - 1$. Thus, we see that if $0 \leq k \leq m - 1$, then the corresponding pole in Z_Φ is cancelled by a zero, so we only have (unmatched) poles in Z_Φ for $k \geq \max\{0, m\}$, or equivalently $m \leq k$.

Next, let us evaluate the contour integral above. First, let us rewrite it as a sum of residues:

$$Z_{nlsm} = \sum_{k=0}^{\infty} \sum_{m \leq k} e^{-im\theta} e^{-4\pi q r} \text{Res}_{\tau=m/2-k} \left\{ e^{4\pi r \tau} \left(\frac{\Gamma(\tau - m/2)}{\Gamma(1 - \tau - m/2)} \right)^8 \left(\frac{\Gamma(1 - 2\tau + m)}{\Gamma(2\tau + m)} \right)^4 \right\}.$$

Now, let us simplify this expression, following [4][appendix A]. First, define ℓ by $m = k - \ell$, so that the expression above becomes

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} e^{-i(k-\ell)\theta} e^{-4\pi q r} \oint \frac{d\epsilon}{2\pi i} \left\{ e^{4\pi r(-\ell/2-k/2+\epsilon)} \left(\frac{\Gamma(\epsilon - k)}{\Gamma(1 + \ell - \epsilon)} \right)^8 \left(\frac{\Gamma(1 + 2k - 2\epsilon)}{\Gamma(-2\ell - 2\epsilon)} \right)^4 \right\}.$$

Then, define $z = \exp(-2\pi r + i\theta)$ and use the identity

$$\Gamma(x) = \frac{\pi}{\sin \pi x} \frac{1}{\Gamma(1 - x)}$$

to write

$$\begin{aligned} Z_{nlsm} &= \oint \frac{d\epsilon}{2\pi i} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} z^\ell \bar{z}^k (z\bar{z})^{q-\epsilon} \pi^4 \frac{(\sin \pi(-2\ell + 2\epsilon))^4}{(\sin \pi(\epsilon - k))^8} \frac{\Gamma(1 + 2k - 2\epsilon)^4}{\Gamma(1 + k - \epsilon)^8} \frac{\Gamma(1 + 2\ell - 2\epsilon)^4}{\Gamma(1 + \ell - \epsilon)^8}, \\ &= \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{q-\epsilon} \pi^4 \frac{(\sin 2\pi\epsilon)^4}{(\sin \pi\epsilon)^8} \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 2k - 2\epsilon)^4}{\Gamma(1 + k - \epsilon)^8} \right|^2, \end{aligned}$$

where the complex conjugation acts only on z , not ϵ .

To evaluate this, first define

$$f(\epsilon) = \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1+2k-2\epsilon)^4}{\Gamma(1+k-\epsilon)^8} \right|^2.$$

Then, it is straightforward to show that

$$\begin{aligned} Z_{nlsm} = \frac{8}{3} (z\bar{z})^q & \left[-\ln(z\bar{z})^3 f(0) - 8\pi^2 f'(0) + 3\ln(z\bar{z})^2 f'(0) \right. \\ & \left. + \ln(z\bar{z}) (8\pi^2 f(0) - 3f''(0)) + f^{(3)}(0) \right]. \end{aligned}$$

Now, in principle, a normalized Z_{nlsm} in a one-parameter model such as this should match [4][equ'n (2.19)]

$$\begin{aligned} \exp(-K) = & -\frac{i}{6} \kappa (t - \bar{t})^3 + \frac{\zeta(3)}{4\pi^3} \chi(X) + \frac{2i}{(2\pi i)^3} \sum_n N_n (\text{Li}_3(q^n) + \text{Li}_3(\bar{q}^n)) \\ & - \frac{i}{(2\pi i)^2} \sum_n N_n (\text{Li}_2(q^n) + \text{Li}_2(\bar{q}^n)) n(t - \bar{t}), \end{aligned} \quad (2)$$

where κ is the hyperplane triple self-intersection, and

$$\text{Li}_k(q) = \sum_{n=1}^{\infty} \frac{q^n}{n^k}, \quad q = \exp(2\pi i t).$$

Now, one of the properties of t is that close to large radius, it should be defined up to a shift by 1, hence one expects

$$t = \frac{\ln z}{2\pi i} + (\text{terms invariant under } z \mapsto ze^{2\pi i}).$$

Hence, the correct normalization can be computed by dividing Z_{nlsm} by the coefficient of $-(i/6)\kappa \ln(z\bar{z})^3/(2\pi i)^3$. In the present case, $\kappa = 16$, hence we should divide Z_{nlsm} by

$$-i(2\pi i)^3 (z\bar{z})^q f(0) = -i(2\pi i)^3 (z\bar{z})^q \left| \sum_{k=0}^{\infty} z^k \frac{\Gamma(1+2k)^4}{\Gamma(1+k)^8} \right|^2$$

to get

$$\begin{aligned} \exp(-K) = & -i \frac{16}{6} \left[\frac{\ln(z\bar{z})^3}{(2\pi i)^3} + \frac{8\pi^2}{(2\pi i)^3} \frac{f'(0)}{f(0)} - \frac{3}{2\pi i} \frac{\ln(z\bar{z})^2}{(2\pi i)^2} \frac{f'(0)}{f(0)} \right. \\ & \left. - \frac{\ln(z\bar{z})}{2\pi i} \left(\frac{8\pi^2}{(2\pi i)^2} - \frac{3}{(2\pi i)^2} \frac{f''(0)}{f(0)} \right) - \frac{1}{(2\pi i)^3} \frac{f^{(3)}(0)}{f(0)} \right]. \end{aligned} \quad (3)$$

Next, we need to solve for $t = t(z)$. To do this, we compare the expression above to equation (2).

If we write

$$t(z) = \frac{\ln z}{2\pi i} + \frac{\Delta(z)}{2\pi i} \quad (4)$$

for some function $\Delta(z)$, then judging from the expression above, the $\ln(z\bar{z})^2$ term (which should only arise in the $(t - \bar{t})^3$ term in $\exp(-K)$) implies that

$$\frac{\Delta + \bar{\Delta}}{2\pi i} = -\frac{1}{2\pi i} \frac{f'(0)}{f(0)} = -\frac{1}{2\pi i} \frac{\partial}{\partial \epsilon} \ln f(\epsilon) \Big|_{\epsilon=0}.$$

Define

$$g(\epsilon) = \sum_{k=0}^{\infty} z^k \frac{\Gamma(1 + 2k - 2\epsilon)^4}{\Gamma(1 + k - \epsilon)^8}$$

so that $f(\epsilon) = |g(\epsilon)|^2$, then

$$\Delta(z) = 2\pi i C - \frac{\partial}{\partial \epsilon} \ln g(\epsilon) \Big|_{\epsilon=0}$$

for C an undetermined real number, or equivalently

$$t(z) = \frac{\ln z}{2\pi i} + C - \frac{1}{2\pi i} \frac{\partial}{\partial \epsilon} \ln g(\epsilon) \Big|_{\epsilon=0}.$$

Exponentiating equation (4), we get

$$\begin{aligned} q &= \exp(2\pi i t) = z e^{2\pi i C} \exp\left(-\frac{\partial}{\partial \epsilon} \ln g(\epsilon) \Big|_{\epsilon=0}\right), \\ &= z e^{2\pi i C} (1 + 64z + 7072z^2 + 991232z^3 + 158784976z^4 + 27706373120z^5 \\ &\quad + 5130309889536z^6 + \mathcal{O}(z^7)), \end{aligned}$$

which can be inverted to find

$$\begin{aligned} z &= q e^{-2\pi i C} - 64q^2 e^{-4\pi i C} + 1120q^3 e^{-6\pi i C} - 38912q^4 e^{-8\pi i C} - 1536464q^5 e^{-10\pi i C} \\ &\quad - 177833984q^6 e^{-12\pi i C} - 19069001216q^7 e^{-14\pi i C} + \mathcal{O}(q^8). \end{aligned}$$

Next, we compare the expressions for e^{-K} in equations (2), (3). In particular, equation (2) contains two different terms with Gromov-Witten invariants, each multiplied by a different power of t . By demanding these two expressions match, we should be able to get two independent expressions for the same Gromov-Witten invariants, which will provide a good consistency check on our computations.

For definiteness, let us turn z 's into t 's, and compare coefficients of various powers of t . Applying equation (4), we find that equation (3) can be rewritten in the form

$$e^{-K} = -i\frac{16}{6} \left[(t - \bar{t})^3 + \frac{(t - \bar{t})}{(2\pi i)^2} \left(3 \left(\frac{\partial}{\partial \epsilon} \right)^2 \ln f(\epsilon) \Big|_{\epsilon=0} - 8\pi^2 \right) - \frac{1}{(2\pi i)^3} \left(\frac{\partial}{\partial \epsilon} \right)^3 \ln f(\epsilon) \Big|_{\epsilon=0} \right].$$

Comparing the expression above to equation (2), we find that, from the coefficient of $(t - \bar{t})$,

$$-\frac{i}{(2\pi i)^2} \sum_n n N_n (\text{Li}_2(q^n) + \text{Li}_2(\bar{q}^n)) = -i\frac{16}{6} \frac{1}{(2\pi i)^2} \left[3 \left(\frac{\partial}{\partial \epsilon} \right)^2 \ln f(\epsilon) \Big|_{\epsilon=0} - 8\pi^2 \right],$$

and from the coefficient of (1),

$$\frac{\zeta(3)}{4\pi^3} \chi(X) + \frac{2i}{(2\pi i)^3} \sum_n N_n (\text{Li}_3(q^n) + \text{Li}_3(\bar{q}^n)) = i\frac{16}{6} \frac{1}{(2\pi i)^3} \left(\frac{\partial}{\partial \epsilon} \right)^3 \ln f(\epsilon) \Big|_{\epsilon=0}.$$

This gives us two separate expressions for the Gromov-Witten invariants N_n ; by using both, we get a good consistency check of our results. From a series expansion, each implies the (same) values, below, for Gromov-Witten invariants:

$$\begin{aligned} N_1 &= 512, \\ N_2 &= 9728, \\ N_3 &= 416256, \\ N_4 &= 25703936, \\ N_5 &= 1957983744, \\ N_6 &= 170535923200, \end{aligned}$$

and constant $C = 0$.

Now, let us compare to existing results. Counts of rational curves in $\mathbb{P}^7[2, 2, 2, 2]$ are listed in *e.g.* [28], [29][p. 36], which have the following results²

Degree	Count
1	512
2	9728
3	416256
4	25703936

² Reference [28] lists the number of degree 1 curves as 512, and in [29][p. 36], we have been informed by one of the authors (S. Hosono) that rational curves of degree d appear in the table as the n_{d+1}^r entry. (Elliptic curves of degree d are listed as n_d^e .)

Thus, we see that we have correctly computed the Gromov-Witten invariants, a good consistency check of this approach.

4 Noncommutative resolution

Now, let us turn to the Landau-Ginzburg phase of the same GLSM, at $r \ll 0$. Let us first derive an expression for the partition function, and then proceed as above to derive analogues of Gromov-Witten invariants. These will form our prediction for Gromov-Witten invariants of the noncommutative resolution, and whether they match the Gromov-Witten invariants of a smooth branched double cover will tell us whether the SCFTs for the noncommutative resolutions may be continuously connected by complex structure deformations to SCFT's for smooth branched double covers.

Our analysis begins with the expression for the partition function in equation (1). Since we are now considering the $r \ll 0$ phase, we will close the contour on the right half-plane. We will assume that \mathfrak{q} is just below $1/2$.

First, we will show that all of the poles of Z_Φ are cancelled out by zeroes of the same. From our previous analysis, recall that poles of the numerator of Z_Φ are located at

$$\tau = m/2 - k$$

for $k \geq 0$, and poles of the denominator are located at

$$\tau = 1 - m/2 + k_1$$

for $k_1 \geq 0$. When $m \geq 1$, the numerator can have poles inside the contour, and this will happen for $k < m/2$. If there is a pole inside the contour for some k , then it is cancelled by a pole of the denominator with

$$k_1 = m - k - 1,$$

which is guaranteed to be nonnegative by the fact that $k < m/2$ and $m \geq 1$.

Now, let us count contributing poles of Z_P . For one of the poles of Z_P to lie inside the integration contour, judging solely from the numerator, we should require $k \geq \max(0, -m)$. However, some of those poles will be cancelled by poles in the denominator of Z_P . Suppose without loss of generality that $m < 0$. If $k_1 = -2m - 1 - k$, then formally (ignoring signs on k, k_1) a pole in the numerator and denominator will coincide. To generate a pole inside the contour, k_1 must satisfy $0 \leq k_1 < -m - 1$, which is equivalent to $-m \leq k < -2m$. Hence, the only contributing poles will have $k \geq \max(0, -2m)$, or equivalently $m \geq -k/2$.

Therefore, for $r \ll 0$, we can write the partition function in equation (1) as

$$\begin{aligned}
Z_{nlsm} &= - \sum_{k=0}^{\infty} \sum_{m \geq -k/2} e^{-im\theta} e^{-4\pi q r} \\
&\quad \text{Res}_{\tau=(1/2)(m+1+k)} \left\{ e^{4\pi r \tau} \left(\frac{\Gamma(\tau - m/2)}{\Gamma(1 - \tau - m/2)} \right)^8 \left(\frac{\Gamma(1 - 2\tau + m)}{\Gamma(2\tau + m)} \right)^4 \right\}, \\
&= - \sum_{k=0}^{\infty} \sum_{m \geq -k/2} e^{-im\theta} e^{-4\pi q r} \oint \frac{d\epsilon}{2\pi i} e^{2\pi r(m+1+k)} e^{4\pi r \epsilon} \\
&\quad \cdot \left(\frac{\Gamma((k+1)/2 + \epsilon)}{\Gamma((1-k)/2 - m - \epsilon)} \right)^8 \left(\frac{\Gamma(-k - 2\epsilon)}{\Gamma(2m + 1 + k + 2\epsilon)} \right)^4,
\end{aligned}$$

where the overall sign takes into account the orientation on the original τ contour. Define $\ell = k + 2m$, then we can write

$$\begin{aligned}
Z_{nlsm} &= - \sum_{\delta=0}^1 \sum_{k, \ell = \delta, 2+\delta, 4+\delta, \dots} e^{-i(\ell-k)\theta/2} e^{-4\pi q r} \oint \frac{d\epsilon}{2\pi i} e^{2\pi r(\ell+k+2)/2} e^{4\pi r \epsilon} \\
&\quad \cdot \left(\frac{\Gamma((k+1)/2 + \epsilon)}{\Gamma((1-\ell)/2 - \epsilon)} \right)^8 \left(\frac{\Gamma(-k - 2\epsilon)}{\Gamma(\ell + 1 + 2\epsilon)} \right)^4.
\end{aligned}$$

Define a, b by

$$k = 2a + \delta, \quad \ell = 2b + \delta,$$

then

$$\begin{aligned}
Z_{nlsm} &= - \sum_{\delta=0}^1 \sum_{a, b=0}^{\infty} e^{-i(b-a)\theta} e^{-4\pi q r} \oint \frac{d\epsilon}{2\pi i} e^{2\pi r(a+b+\delta+1)} e^{4\pi r \epsilon} \\
&\quad \cdot \left(\frac{\Gamma(a + (\delta+1)/2 + \epsilon)}{\Gamma(-b + (1-\delta)/2 - \epsilon)} \right)^8 \left(\frac{\Gamma(-2a - \delta - 2\epsilon)}{\Gamma(2b + \delta + 1 + 2\epsilon)} \right)^4.
\end{aligned}$$

Define $z = \exp(-2\pi r + i\theta)$ as at large radius, then

$$\begin{aligned}
Z_{nlsm} &= - \sum_{\delta=0}^1 \sum_{a, b=0}^{\infty} z^{-b} \bar{z}^{-a} \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{q-(\delta+1)/2-\epsilon} \\
&\quad \cdot \left(\frac{\Gamma(a + (\delta+1)/2 + \epsilon)}{\Gamma(-b + (1-\delta)/2 - \epsilon)} \right)^8 \left(\frac{\Gamma(-2a - \delta - 2\epsilon)}{\Gamma(2b + \delta + 1 + 2\epsilon)} \right)^4, \\
&= - \sum_{\delta=0}^1 \sum_{a, b=0}^{\infty} z^{-b} \bar{z}^{-a} \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{q-(\delta+1)/2-\epsilon} \pi^{-4} \\
&\quad \cdot \frac{[\sin \pi((\delta-1)/2 + \epsilon)]^8}{[\sin \pi(\delta + 2\epsilon)]^4} \frac{\Gamma(a + (1+\delta)/2 + \epsilon)^8}{\Gamma(1 + 2a + \delta + 2\epsilon)^4} \frac{\Gamma(b + (1+\delta)/2 + \epsilon)^8}{\Gamma(1 + 2b + \delta + 2\epsilon)^4}.
\end{aligned}$$

Proceeding as at large radius, define

$$f_\delta(\epsilon) = \left| \sum_{m=0}^{\infty} \left(\frac{1}{z} \right)^m \frac{\Gamma(m + (1 + \delta)/2 + \epsilon)^8}{\Gamma(2m + 1 + \delta + 2\epsilon)^4} \right|^2,$$

where the complex conjugation acts only on z . Then we can write

$$\begin{aligned} Z_{nlsm} &= - \sum_{\delta=0}^1 \oint \frac{d\epsilon}{2\pi i} (z\bar{z})^{q-(\delta+1)/2-\epsilon} \pi^{-4} \frac{[\sin \pi((\delta-1)/2 + \epsilon)]^8}{[\sin \pi(\delta + 2\epsilon)]^4} f_\delta(\epsilon), \\ &= -(z\bar{z})^{q-1/2} \frac{1}{96\pi^8} \left(-f_0(0) \ln(z\bar{z})^3 - 8\pi^2 f'_0(0) + 3 \ln(z\bar{z})^2 f'_0(0) \right. \\ &\quad \left. + \ln(z\bar{z}) (8\pi^2 f_0(0) - 3f''_0(0)) + f_0^{(3)}(0) \right). \end{aligned}$$

The only contribution to the residue is from $\delta = 0$, which at some level is a result of the fact that at large radius we have a complete intersection of quadrics. For a more general case, one expects that some contributions from $\delta \neq 0$ might be nonzero, which would impair our ability to apply [4] to make predictions for Gromov-Witten invariants.

To extract the mirror map, we need to find the triple self-intersection κ . For a smooth branched double cover, $\kappa = 2$, essentially because it is a double cover of \mathbf{P}^3 – κ counts the number of elements in the cover, effectively.. In the present case, we want the analogue of κ for a noncommutative resolution of a singular branched double cover. We do not know how to define κ in general for such; however, the triple self-intersection can be computed away from the location of the singularities, so we will assume $\kappa = 2$ for the noncommutative resolution also.

Proceeding as at large-radius, we should normalize Z_{nlsm} so that it matches $\exp(-K)$, which contains a

$$-\frac{i}{6} \kappa (t - \bar{t})^3$$

term. As at large-radius, because of B field shifts, t should have the form

$$t = \frac{\ln z}{2\pi i} + (\text{terms invariant under } z \mapsto ze^{2\pi i})$$

and the term above should be the only possible source of a $\ln(z\bar{z})^3$ term. Hence, the correct normalization should be obtained by dividing Z_{nlsm} by

$$\frac{(z\bar{z})^{q-1/2}}{(i/6)(2)} \frac{1}{96\pi^8} (-) f_0(0) (2\pi i)^3,$$

which yields

$$\begin{aligned} e^{-K} &= -\frac{i}{6}(2) \left(\frac{\ln(z\bar{z})^3}{(2\pi i)^3} + \frac{8\pi^2}{(2\pi i)^3} \frac{f'_0(0)}{f_0(0)} - \frac{3}{2\pi i} \frac{\ln(z\bar{z})^2}{(2\pi i)^2} \frac{f'_0(0)}{f_0(0)} \right. \\ &\quad \left. - \frac{\ln(z\bar{z})}{2\pi i} \left(\frac{8\pi^2}{(2\pi i)^2} - \frac{3}{(2\pi i)^2} \frac{f''_0(0)}{f_0(0)} \right) - \frac{1}{(2\pi i)^3} \frac{f_0^{(3)}(0)}{f_0(0)} \right). \end{aligned}$$

If we write

$$t(z) = \frac{\ln z}{2\pi i} + \frac{\Delta(z)}{2\pi i}$$

then

$$(t - \bar{t})^3 = \frac{\ln(z\bar{z})^3}{(2\pi i)^3} + 3 \left(\frac{\Delta + \bar{\Delta}}{2\pi i} \right) \frac{\ln(z\bar{z})^2}{(2\pi i)^2} + \dots$$

hence we read off that

$$\begin{aligned} \Delta + \bar{\Delta} &= -\frac{f'_0(0)}{f_0(0)} \\ &= -\frac{\partial}{\partial \epsilon} \ln g(\epsilon) \Big|_{\epsilon=0} + \text{c.c.}, \end{aligned}$$

where

$$g(\epsilon) = \sum_{m=0}^{\infty} \left(\frac{1}{z} \right)^m \frac{\Gamma(m + 1/2 + \epsilon)^8}{\Gamma(2m + 1 + 2\epsilon)^4}.$$

This implies

$$q \equiv \exp(2\pi i t) = z e^{2\pi i C} \exp \left(-\frac{\partial}{\partial \epsilon} \ln g(\epsilon) \Big|_{\epsilon=0} \right),$$

or more simply

$$\begin{aligned} q &= z e^{2\pi i C} \left(65536 - 64 \frac{1}{z} - \frac{93}{2048} \frac{1}{z^2} - \frac{85}{1048576} \frac{1}{z^3} - \frac{3251101}{17592186044416} \frac{1}{z^4} \right. \\ &\quad \left. - \frac{8596595}{18014398509481984} \frac{1}{z^5} + \mathcal{O} \left(\frac{1}{z^6} \right) \right), \end{aligned}$$

where C is an undetermined real constant. Inverting, we find

$$\begin{aligned} \frac{1}{z} &= 65536 q^{-1} e^{2\pi i C} - 4194304 q^{-2} e^{4\pi i C} + 73400320 q^{-3} e^{6\pi i C} - 2550136832 q^{-4} e^{8\pi i C} \\ &\quad - 100693704704 q^{-5} e^{10\pi i C} - 11654527975424 q^{-6} e^{12\pi i C} + \mathcal{O}(q^{-7}). \end{aligned}$$

Now, proceeding as before, after algebra we can write

$$\begin{aligned} e^{-K} &= -\frac{i}{3} \left[(t - \bar{t})^3 + \frac{(t - \bar{t})}{(2\pi i)^2} \left(3 \left(\frac{\partial}{\partial \epsilon} \right)^2 \ln f_0(\epsilon) \Big|_{\epsilon=0} - 8\pi^2 \right) \right. \\ &\quad \left. - \frac{1}{(2\pi i)^3} \left(\frac{\partial}{\partial \epsilon} \right)^3 \ln f_0(\epsilon) \Big|_{\epsilon=0} \right]. \end{aligned}$$

Comparing with equation (2), we find that

$$\sum_n n N_n (\text{Li}_2(q^{-n}) + \text{Li}_2(\bar{q}^{-n})) = \frac{1}{3} \left(3 \left(\frac{\partial}{\partial \epsilon} \right)^2 \ln f_0(\epsilon) \Big|_{\epsilon=0} - 8\pi^2 \right)$$

and

$$\frac{\zeta(3)\chi(2\pi i)^3}{4\pi^3 i} + 2 \sum_n N_n (\text{Li}_3(q^{-n}) + \text{Li}_3(\bar{q}^{-n})) = \frac{1}{3} \left(\frac{\partial}{\partial \epsilon} \right)^3 \ln f_0(\epsilon) \Big|_{\epsilon=0}$$

(where for obvious reasons we have replaced q with q^{-1}).

By expanding each in series, one can extract predictions for Gromov-Witten invariants, and doing so for both equations above gives a good consistency check. One finds, for both of the equations, that the Gromov-Witten invariants are given by

$$\begin{aligned} N_1 &= 64, \\ N_2 &= 1216, \\ N_3 &= 52032, \\ N_4 &= 3212992, \\ N_5 &= 244747968, \\ N_6 &= 21316990400, \\ N_7 &= 2037544347200, \\ N_8 &= 208507887048384, \\ N_9 &= 22480719508041216, \end{aligned}$$

with constant $C = 0$.

Now, let us compare to known results for generic smooth branched double covers of \mathbb{P}^3 . Such smooth cases can be described as hypersurfaces of the form

$$y^2 = f_8(x_1, \dots, x_4),$$

which is to say, $\mathbb{P}_{[1,1,1,1,4]}^5[8]$, and are discussed in *e.g.* [30]; table 3 in that reference lists

Degree	Count
0	2
1	29504
2	128834912
3	1423720545880
4	23193056024793312

Thus, we see that the noncommutative resolutions of branched double covers appearing in the GLSM for $\mathbb{P}^7[2, 2, 2, 2]$, cannot be continuously connected in SCFT moduli space by complex structure deformations to smooth branched double covers, as the Gromov-Witten invariants are demonstrably different.

Given this physics computation, it is also natural to ask to what mathematics this computation corresponds. Given the mathematical definition of noncommutative resolutions in

terms of sheaf theory, it is not clear to the author how one would go about directly defining Gromov-Witten invariants mathematically – perhaps these invariants are encoding information about the CFT itself, rather than the noncommutative structure *per se*. However, there might be an indirect method³. Although a direct definition of Gromov-Witten invariants seems obscure, a direct definition of Donaldson-Thomas invariants for such noncommutative resolutions seems much more nearly straightforward. One might then be able to use the Donaldson-Thomas/Gromov-Witten correspondence to formally define a set of integers, which would play the same role as Gromov-Witten invariants, and might reasonably be called Gromov-Witten invariants of a noncommutative resolution. Perhaps the numbers we have computed could be obtained in this fashion. We will leave such conjectured definitions to future work.

5 Conclusions

In this paper, we have applied the recent GLSM localization techniques of [4] to compute the Gromov-Witten invariants of an abstract CFT realizing a noncommutative resolution (in Kontsevich’s sense) of a singular branched double cover. As those invariants do not match those of related smooth branched double covers, we conclude that they cannot be related by complex structure deformations in the (2,2) SCFT moduli space.

It would also be interesting to apply the methods of [4] to understand analogues of Gromov-Witten invariants for theories close to Landau-Ginzburg orbifolds, as described in *e.g.* [31, 32]. In particular, invariants for the Landau-Ginzburg point of the GLSM for the quintic in \mathbb{P}^4 were computed in [33], and it would be interesting to rederive them using localization methods in GLSM’s. The methods described here are not directly applicable: for example, the Landau-Ginzburg point of the quintic does not have a B field, so there is no analogue of the statement that t should contain a term proportional to $\ln z$, and indeed the partition function at the Landau-Ginzburg point does not contain terms involving $\ln z$ ’s. Nevertheless, if a method could be found, the derivation would be interesting.

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³We would like to thank T. Pantev for suggesting this.

References

- [1] A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E. Sharpe, “Non-birational twisted derived equivalences in abelian GLSMs,” *Comm. Math. Phys.* **294** (2010) 605-645, [arXiv: 0709.3855](#).
- [2] N. Seiberg, E. Witten, “String theory and noncommutative geometry,” [hep-th/9908142](#), *JHEP* 9909 (1999) 032.
- [3] D. Roggenkamp, K. Wendland, “Limits and degenerations of unitary conformal field theories,” [hep-th/0308143](#), *Comm. Math. Phys.* **251** (2004) 589-643.
- [4] H. Jockers, V. Kumar, J. Lapan, D. Morrison, M. Romo, “Two-sphere partition functions and Gromov-Witten invariants,” [arXiv: 1208.6244](#).
- [5] K. Hori, D. Tong, “Aspects of non-abelian gauge dynamics in two-dimensional $N = (2, 2)$ theories,” *JHEP* 0705 (2007) 079, [hep-th/0609032](#).
- [6] R. Donagi, E. Sharpe, “GLSM’s for partial flag manifolds,” *J. Geom. Phys.* **58** (2008) 1662-1692, [arXiv: 0704.1761](#).
- [7] K. Hori, “Duality in two-dimensional $(2,2)$ supersymmetric non-abelian gauge theories,” [arXiv: 1104.2853](#).
- [8] H. Jockers, V. Kumar, J. Lapan, D. Morrison, M. Romo, “Nonabelian 2d gauge theories for determinantal Calabi-Yau varieties,” [arXiv: 1205.3192](#).
- [9] N. Addington, E. Segal, E. Sharpe, “D-brane probes, branched double covers, and noncommutative resolutions,” [arXiv: 1211.2446](#).
- [10] A. Kuznetsov, “Homological projective duality,” [math.AG/0507292](#), *Publ. Math. Inst. Hautes Études Sci.* **105** (2007) 157-220.
- [11] A. Kuznetsov, “Derived categories of quadric fibrations and intersections of quadrics,” [math.AG/0510670](#), *Adv. Math.* **218** (2008) 1340-1369.
- [12] A. Kuznetsov, “Homological projective duality for Grassmannians of lines,” [math.AG/0610957](#).
- [13] M. Ballard, D. Favero, L. Katzarkov, “Variation of geometric invariant theory quotients and derived categories,” [arXiv: 1203.6643](#).
- [14] E. Sharpe, “Derived categories and stacks in physics,” contribution to the proceedings of the ESI research conference on homological mirror symmetry (Vienna, Austria, June 2006), [hep-th/0608056](#).

- [15] E. Sharpe, “Landau-Ginzburg models, gerbes, and Kuznetsov’s homological projective duality,” pp. 237-249 in *Superstrings, geometry, topology, and C^* algebras*, ed. R. Doran, G. Friedman, J. Rosenberg, Proc. Symp. Pure Math. **81**, Amer. Math. Soc., Providence, RI, 2010.
- [16] E. Sharpe, “GLSMs, gerbes, and Kuznetsov’s homological projective duality,” contribution to the proceedings of *Quantum theory and symmetries 6*, [arXiv: 1004.5388](#).
- [17] M. Van den Bergh, “Non-commutative quadrics,” [arXiv: 0807.3753](#).
- [18] M. Van den Bergh, “Three dimensional flops and noncommutative rings,” *Duke Math. J.* **122** (2004) 423-455.
- [19] M. Van den Bergh, “Non-commutative crepant resolutions,” pp. 749-770 in *The legacy of Niels Henrik Abel; the Abel bicentennial (Oslo, 2002)*, ed. O. Laudal, R. Piene, Springer-Verlag, Berlin, 2004, [math/0211064](#).
- [20] R.-O. Buchweitz, G. Leuschke, M. Van den Bergh, “Non-commutative desingularizations of determinantal varieties, I,” [arXiv: 0911.2659](#).
- [21] H. Dao, “Remarks on non-commutative crepant resolutions of complete intersections,” [arXiv: 0911.4541](#).
- [22] D. Berenstein, R. G. Leigh, “Resolution of stringy singularities by noncommutative algebras,” *JHEP* **0106** (2001) 030, [hep-th/0105229](#).
- [23] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe, M. Ando, “Cluster decomposition, T-duality, and gerby CFT’s,” [hep-th/0606034](#), *Adv. Theor. Math. Phys.* **11** (2007) 751-818.
- [24] J. Gomis, S. Lee, “Exact Kähler potential from gauge theory and mirror symmetry,” [arXiv: 1210.6022](#).
- [25] J. Distler, “Notes on N=2 sigma models,” pp. 234-256 in *String theory and quantum gravity ’92* (Trieste 1992), [hep-th/9212062](#).
- [26] F. Benini, S. Cremonesi, “Partition functions of $N = (2, 2)$ gauge theories on S^2 and vortices,” [arXiv: 1206.2356](#).
- [27] N. Doroud, J. Gomis, B. Le Floch, S. Lee, “Exact results in $D = 2$ supersymmetric gauge theories,” [arXiv: 1206.2606](#).
- [28] S. Katz, “Lines on complete intersection threefolds with $K = 0$,” *Math. Z.* **191** (1986) 293-296.
- [29] S. Hosono, A. Klemm, S. Theisen, S.-T. Yau, “Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces,” *Nucl. Phys.* **B433** (1995) 501-554, [hep-th/9406055](#).

- [30] D. Morrison, “Picard-Fuchs equations and mirror maps for hypersurfaces,” pp. 185-199 in *Mirror Symmetry I*, ed. S.-T. Yau, Amer. Math. Soc, International Press, 1998.
- [31] H. Fan, T. Jarvis, Y. Ruan, “The Witten equation, mirror symmetry and quantum singularity theory,” **arXiv:** 0712.4021.
- [32] H. Fan, T. Jarvis, Y. Ruan, “The Witten equation and its virtual fundamental cycle,” **arXiv:** 0712.4025.
- [33] A. Chiodo, Y. Ruan, “Landau-Ginzburg / Calabi-Yau correspondence for quintic three-folds via symplectic transformations,” **arXiv:** 0812.4660.